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Annex No. 10 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University	
Applicant	Ioannis Chrysikos
Habilitation thesis	G-structures, Dirac operators with torsion and special spinor fields
Reviewer	Aleksy Tralle, Professor, Dr. habil.
Reviewer's home unit, institution	Department of Maths&Comp. Sci., University of Warmia and Mazury in Olsztyn, Poland

[Review text] The review is attached as separate file.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

1. Describe your results on classifying invariant connections on isotropy irreducible homogeneous spaces

Conclusion

The habilitation thesis entitled "G-structures, Dirac operators with torsion and special spinor fields" by Ioannis Chrysikos **fulfils** requirements expected of a habilitation thesis in the field of Mathematics – Geometry.

Date: 03.04.2024 Signature: Aleksy Tralle

REVIEW OF THE HABILITATION THESIS
BY IOANNIS CHRYSIKOS
G-STRUCTURES, DIRAC OPERATORS WITH TORSION AND SPECIAL SPINOR FIELDS

Ioannis Chrysikos habilitation thesis describes the results obtained by the candidate, solely or with co-authors and published in 5 strong papers in good mathematical journals *Annals of Global Analysis and Geometry (2016), Advances in Applied Clifford Algebras (2017) , Journal of Geometry and Physics (2019), International Journal of Mathematics (2020)* and *Annali di Matematica Pura ed Applicata (2021).* These works have essential impact on the development of the whole area of metric connections with skew torsion and related areas of mathematical physics. Therefore, these results constitute a high achievement in differential geometry. I would divide this collection into two parts, the first one consisting of papers A,B (in the notation used in the thesis), and the second of C,D, E since they differ by the techniques and considered classes of Riemannian manifolds. The contents of the papers is well explained in the thesis, therefore, I will analyze in more detail papers A and ,as samples.

1. Spinor part

The candidate is interested in the following equations which are very important in mathematical physics (here (M, g) is a compact Riemannian n-manifold):

$$\nabla^s_X \varphi = \zeta X \cdot \varphi, \, \forall X \in \Gamma(TM)$$

(KIlling spinor equation)

$$\nabla^s_X \varphi + \frac{1}{n} X \cdot D^s(\varphi), \forall X \in \Gamma(TM)$$

(twistor spinor equation),
where $\varphi \in \Gamma(\Sigma)$ is a section of a spinor bundle. These equations are determined by a metric connection ∇^s with skew torsion $4sT$ related to the Levi-Civita connection via equation

$$g(\nabla^s_X Y, Z) + 2sT(X, Y, Z).$$

Under the above assumptions on ∇^s, T acts on the spinor bundle Σ and one obtains the decomposition $\Sigma = \oplus_\gamma \Sigma_\gamma$, $\gamma \in \operatorname{Spec} T$. Under these assumptions Chrysikos proves the following general and important result.

Theorem 1. *Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and assume that $\varphi \in \Gamma(\Sigma)$ is a spinor field parallel with respect to ∇^c, where $\nabla^c = \nabla^g + \frac{1}{2}T$ is the characteristic connection. Let $\gamma \in \mathbb{R} \setminus \{0\}$. Then the following conditions are equivalent:*

(1) $\varphi \in \Gamma(\Sigma_\gamma) \cap \mathrm{Ker}(P^c)$ *with respect to the family* $\{\nabla^c, s \neq \frac{1}{4}\}$,
(2) φ *is a Killing spinor with respect to* ∇^s *and with* $\zeta = 3(1-4s)\frac{\gamma}{4n}$,
(3) φ *is a Riemannian Killing spinor with* $k = 3\frac{\gamma}{4}$.

This theorem is quite general and enables one to better understand the equations on several very important objectcs: on 6-dimensional nearly Kähler manifolds, nearly parallel G_2-manifolds, as well as find geometric constraints (for example, expressed via Einstein and ∇-Einstein conditions). Paper A contains several other important observations and examples. To summarise, this is a high level research extending our knowledge on Killing spinor and twistor spinor equations.

Paper B is written in the same style and derives a generalized $\frac{1}{2}$-Ricci type formula for the spinorial action of the Ricci endomorphism. This formula clarifies and sheds new light on some classical results related to Dirac operator. This paper is also very good and extends our knowledge in this area.

2. Invariant connections part

The second part of the thesis is devoted to homogeneous spaces. In this case, Ioannis is still inspired by part I, but adds invariance conditions, and, generally speaking, uses methods of Lie Theory and representation theory. Let me concentrate on the brilliant paper C, which I particularly like.

Invariant connections on principal U-bundels over a homogeneous space G/K are in one to one correspondence with linear maps $\Lambda : \mathfrak{g} \to \mathfrak{u}$ satisying some equivariance condition. It is well known, but it is also known that this description is not easy to use in practice, and it is challenging to understand the set of such maps in general. On the other hand, for symmetric spaces, Laquer was able to show that this set is quite small: either consists only of canonical cannection (coniciding in the symmetric case with Levi-Civita connection), or of 1-dimensional family (for example, in the case $SU(n)/SO(n)$). The role of finding a good description of the set of invariant connections in Lie theoretic terms is clear - applications are numerous in any problem involving invariant geometric structures on homogeneous spaces. Ioannis Chrysikos with co-authors did a very good job considering this problem and developing a new and effective approach to the problem in the case of reductive homogeneous spaces, that is G/K such that there is a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}.$$

The approach developed in the analyzed paper is based on intepreting the space of invariant connections as a space of K-intertwining maps $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Next, several submodules of this module are interpreted geometrically. For example, the space of homogeneous metric connections is

$$\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}).$$

Working with some decompositions of the module $\otimes^3\mathfrak{m}$ and their Young diagrams, the authors calculate the defect $\epsilon = N_{mtr} - l$, where N_{mtr} denotes the dimension of the space of metric connections, while l denotes the space of invariant connections which have the same geodesics as the Levi-Civita connection. This dimension is expressed in terms of $(2, 1)$-plethysm of the K-representation $\mathfrak{m}$. I consider this approach to the problem as pioneering in many respects. I will not cite all the results of this nice paper, however, I mention that the developed methods enabled the authors to partially classify invariant connections on non-symmetric isotropy irreducible homogeneous spaces. This class of spaces is of particular interest in differential geometry. The results are given in Tables 2, 3 and 4 of the paper. These tables contain new G-invariant metric connections and the dimesnions of the spaces of invariant connections, sometimes unexpectedly large (for example, 8). To sum up, I consider the methods developed in this paper as very promising for future research (also in other directions involving homogeneous geometric structures).

In some sense, paper D uses methods which have some similarities to paper B. All canonical presentations G/H of 8-dimensional homogeneous manifolds with $spin(7)$-structures are found.

Paper E classifies algebraic types of $SO^*(2n)$ or $SO^*(2n)Sp(1)$ structures and develop a procedure that allows for treating them as the symplectic analogue of the hypercomplex/quaternionic Hermitian geometries.

3. Conclusion

Th contribution of Ioannis Chrysikos to several areas of geometric research are very substantial:

(1) much better understanding of spinor and twistor spinor equations under the parallel condition with respect to connections with skew torsion, together with a new insights in the models of Type II in string theory,
(2) the first essential and breakthrough results on classification of invariant connections on (classes of) homogeneous spaces.

The habilitation thesis of Ioannis Chrysikos fulfills all the requirements and proves that the author is a high level mathematician who made essential contributions to differential geometry.